Eigenstate Preparation by Phase Decoherence

Rolando Somma
Perimeter Institute
Ontario, Canada
rsomma@perimeterinstitute.ca

Sergio Boixo
California Institute of Technology
California, USA
boixo@caltech.edu

Emanuel Knill
National Institute of
Standards and Technology
Colorado, USA
knill@boulder.nist.gov

***Abstract*—A computation in adiabatic quantum computing is implemented by traversing a path of nondegenerate eigenstates of a continuous family of Hamiltonians. We introduce a method that traverses a discretized form of the path: at each step we apply the instantaneous Hamiltonian for a random time. The resulting decoherence approximates a projective measurement onto the desired eigenstate, achieving a version of the quantum Zeno effect. The average cost of our method is $\mathcal{O}(L^2/\Delta)$ for constant error probability, where L is the length of the path of eigenstates and Δ is the minimum spectral gap of the Hamiltonian. For many cases of interest, L does not depend on Δ so the scaling of the cost with the gap is better than the one obtained in rigorous proofs of the adiabatic theorem. We give an example where this situation occurs.**

I. Introduction

In adiabatic quantum computing (AQC) [1] the quantum computation is performed by smoothly changing the interaction parameters of the Hamiltonian under which the system evolves. The initial state is a nondegenerate eigenstate of the Hamiltonian. The adiabatic theorem of quantum mechanics asserts that if the continuously related eigenstates remain nondegenerate and the Hamiltonians change sufficiently slowly, then the final state of the system is close to the continuously related eigenstate of the final Hamiltonian [2]. The last step is a standard projective measurement. AQC is polynomially equivalent to the quantum circuit model [3].

In this paper we give a method for traversing eigenstate paths of Hamiltonians $\{H(s), s \in [0,1]\}$ that differs from AQC by the use of evolution randomization to implement a version of the quantum Zeno effect. We choose a discretization of the eigenstate path and apply the Hamiltonian corresponding to each point for a random time. This simulates a projective measurement onto the instantaneous (non-degenerate) eigenstate basis. If the length between consecutive states, corresponding to consecutive points in the discretization, is small enough, the desired eigenstate can be prepared with sufficiently high probability.

The algorithmic complexity of the randomization method is given by the number of points in the discretization times the complexity of simulating each projective measurement. The latter is given by the average of the absolute evolution time. For error ϵ we obtain a complexity $\mathcal{O}(L^2/(\epsilon\Delta))$, with

$$L = \int_0^1 \parallel |\dot{\psi}(s)\rangle \parallel ds \tag{1}$$

being the length of the path of non-degenerate eigenstates $|\psi(s)\rangle$ of $H(s)$, and $|\dot{\psi}(s)\rangle = \partial_s |\psi(s)\rangle$. (With no loss of generality we assume $\langle\psi(s)|\partial_s\psi(s)\rangle = 0$.) Δ is the minimum spectral gap of the Hamiltonians. The scaling of the cost with the gap is optimal and is better than the $1/\Delta^3$ of rigorous proofs of the adiabatic theorem [4], [5], [6], particularly if L does not depend on Δ. Making the dependence of the cost on L and Δ explicit is important to show quantum speedups for certain problems. An example of a polynomial quantum speedup using the evolution randomization technique was given for the simulated annealing algorithm in Ref. [7].

The paper is organized as follows. In Sec. II we explain how the quantum Zeno effect can be exploited and show how to simulate projective measurement operations by means of evolution randomization. In Sec. III we explain our randomization method. In Sec. IV we give an example where L is independent of Δ. We summarize in Sec. V.

II. Randomized evolutions

The randomization algorithm takes as input a non-degenerate ground state $|\psi(0)\rangle$ of $H(0)$, a continuous and differentiable family of Hamiltonians $H(s)$ having non-degenerate ground states, and aims to output $|\psi(1)\rangle$, the desired ground state of $H(1)$, with bounded error probability. This is done by evolving with $H(s)$ for random time. The randomization method can still be used to prepare any other non-degenerate eigenstate as long as the assumptions we make below are still valid.

A. Adiabatic quantum computing using the Zeno effect

The quantum Zeno effect is based on the fact that, for a small displacement δ', the probability of projecting $|\psi(s+\delta')\rangle$ onto $|\psi(s)\rangle$ decreases with $(\delta')^2$, while the distance between states is linear in δ' [8], [9], [10]. Therefore, for the path of states $\{|\psi(s)\rangle\}$, the final state $|\psi(1)\rangle$ can be prepared from the initial state $|\psi(0)\rangle$ with high fidelity via a sequence of measurement projections onto intermediate states $|\psi(s_1)\rangle, \cdots, |\psi(s_q)\rangle$, $0 < s_1 < \cdots < s_q = 1$. We choose s_j so that the fidelity of the final state with respect to $|\psi(1)\rangle$ is sufficiently close to unity. It is not necessary to keep track of the measurement results at intermediate steps. We assume a parametrization $s(l)$, with $l \in [0, L']$, monotonically increasing in l, $s(0) = 0$ and $s(L') = 1$. We define $|\tilde{\psi}(l)\rangle = |\psi(s(l))\rangle$. (Objects with a tilde correspond to objects in the new parametrization). Later we consider $s(l)$ so that $L' = L$, the path length of Eq. (1). We formulate the

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Zeno method for quantum state preparation as follows [11], [12], [7]:

Lemma 1 (Zeno effect): Consider the continuous path of states $\{|\tilde{\psi}(l)\rangle\}_{l\in[0,L']}$ and assume that, for fixed d and all δ,

$$|\langle\tilde{\psi}(l)|\tilde{\psi}(l+\delta)\rangle|^2 \geq 1-\delta^2 \ . \tag{2}$$

Define the projective-measurement operations onto $|\tilde{\psi}(l)\rangle$ as

$$M_l(\rho) = P_l\rho P_l + \mathcal{E}((\mathbf{1}-P_l)\rho(\mathbf{1}-P_l)) \ , \tag{3}$$

with $P_l = |\tilde{\psi}(l)\rangle\langle\tilde{\psi}(l)|$ and $\mathcal{E}$ arbitrary quantum operations that may vary with l. Then the state $|\tilde{\psi}(L')\rangle$ can be prepared from $|\tilde{\psi}(0)\rangle$ with fidelity $p>0$ by $\lceil (L')^2/(1-p)\rceil$ intermediate projective-measurement operations.

Proof: Divide $[0,L']$ into $q=\lceil (L')^2/(1-p)\rceil$ segments and set $\delta = L'/q$. At every point $l_j = j\delta$, $1\leq j\leq q$, we perform a projective-measurement operation onto $|\tilde{\psi}(l_j)\rangle$. The final state is $M_{l_q}\circ M_{l_{q-1}}\circ\cdots\circ M_{l_1}(\rho)$, with $\rho = |\tilde{\psi}(0)\rangle\langle\tilde{\psi}(0)|$. The output fidelity is bounded as

$$\begin{aligned}
&\mathrm{tr}\,[P_{l_q}(M_{l_q}\circ\cdots\circ M_{l_1}(\rho))] \geq \|P_{l_q}\cdots P_{l_1}|\tilde{\psi}(0)\rangle\|^2 \\
&= \Pi_{j=1}^{q}|\langle\tilde{\psi}(l_j)|\tilde{\psi}(l_{j-1})\rangle|^2 \\
&\geq (1-\delta^2)^q \geq 1-L'^2/q \geq 1-(1-p) = p \ .
\end{aligned} \tag{4}$$

Using Lemma 1 and assuming a *uniform* parametrization, where $L(s(l)) = \tilde{L}(l) = l$ and $L' = L$, we see that the state $|\psi(1)\rangle$ can be obtained with fidelity p starting from $|\psi(0)\rangle$ with $\mathcal{O}(L^2/(1-p))$ projective-measurement operations. In many cases, the uniform parametrization cannot be easily obtained. In these cases we consider a subuniform parametrization $s'(l)$ that can be used to overestimate the path length. To obtain $s'(l)$ from our assumptions we need the following Lemma:

Lemma 2: Suppose that $H(s)$ is differentiable and $\{|\psi(s)\rangle\}$ is a path of nondegenerate ground states of $\{H(s)\}$ with spectral gap $\Delta(s)>0$. Then

$$\|\ |\dot{\psi}(s)\rangle\ \| \leq \frac{\|\ \dot{H}(s)\ \|}{\Delta(s)} \ .$$

Proof: Because $H(s)$ is differentiable, it follows that $|\psi(s)\rangle$ is differentiable. Deriving the eigenvalue equation

$$H(s)|\psi(s)\rangle = E(s)|\psi(s)\rangle \ , \tag{5}$$

we get

$$\langle\psi_j(s)|\dot{\psi}(s)\rangle = \frac{\langle\psi_j(s)|\,\partial_s H(s)|\psi(s)\rangle}{E(s)-E_j(s)} \ , \tag{6}$$

where $|\psi_j(s)\rangle$, $j\in\{2,\ldots,d\}$, is the j-th eigenstate of $H(s)$ (orthogonal to $|\psi(s)\rangle$), having eigenvalue $E_j(s)$.

Without loss of generality $\langle\dot{\psi}(s)\,|\psi(s)\rangle = 0$. This gives

$$\begin{aligned}
\|\ |\partial_s\psi(s)\rangle\ \|^2 &= \sum_{j\geq 2}\frac{|\ \langle\psi_j(s)|\,\partial_s H(s)|\psi(s)\rangle|^2}{|E(s)-E_j(s)|^2} \\
&\leq \frac{\|\ \dot{H}(s)\ \|^2}{\Delta(s)^2} \ .
\end{aligned}$$

Define $\|\dot{H}\| = \sup\|\partial_s H(s)\|$ and $0<\Delta\leq\inf_s\Delta(s)$. We obtain

$$L = \int_0^1 \|\ |\dot{\psi}(s)\rangle\|ds \leq L' = \frac{\|\dot{H}\|}{\Delta} \ . \tag{7}$$

with Δ a lower bound to the minimum absolute value of the gap. This L' is achieved for the parametrization

$$s'(l) = \frac{\Delta}{\|\dot{H}\|}l \ , \tag{8}$$

which is subuniform.

B. Approximating projective-measurement operations through randomized evolutions

As in AQC, we assume that evolutions under $H(s)$ for time t can be implemented at a cost linear in $|t|\|H(s)\|$. That is, we do not take into account the cost of simulating $H(s)$ for small time intervals. By rescaling $H(s)$ if necessary, we can assume that $\|H(s)\|\leq 1$. Thus, the cost of the randomization method is determined by the sum of the absolute evolution times. We consider the case where the evolution time t can be negative.

According to Lemma 1, the Zeno method does not require that we keep track of intermediate measurement results. Thus, any purely dephasing mechanism in the instantaneous eigenbasis of $\tilde{H}(l)$ implements M_l. A natural choice for such a decoherence mechanism is the evolution induced by $\tilde{H}(l)$ for a (unknown) random time $t\in\mathbf{R}$. This is the subject of next theorem, where we bound the residual coherences in terms of the characteristic function of the random time distribution.

Theorem 1 (Randomized dephasing): Let $|\tilde{\psi}(l)\rangle$ be a nondegenerate eigenstate of $\tilde{H}(l)$, and $\{\omega_j\}$ be the energy differences to the other eigenstates $|\tilde{\psi}_j(l)\rangle$. Let T be a random variable associated with the time of evolution under $\tilde{H}(l)$, and $\mathcal{R}_l^T$ the corresponding quantum operation. Then there exists a quantum operation $\mathcal{E}$ such that, for all states ρ,

$$\|(M_l^{\mathcal{E}}-\mathcal{R}_l^T)(\rho)\|_{\mathrm{tr}} \ \leq \epsilon = \sup_{\omega_j}|\Phi(\omega_j)| \ ,$$

where $M_l^{\mathcal{E}}$ is the projective-measurement operation defined in Lemma 1 with $\mathcal{E}$ specified, and Φ is the characteristic function of T.

Proof: We only give an outline of the proof. A detailed proof can be found in Ref. [13]. The main ingredient is that the coherences between the desired (ground) eigenstate and other orthogonal states, after the randomized evolution, are given by

$$\mathcal{R}_l^T(|\tilde{\psi}(l)\rangle\langle\tilde{\psi}_j(l)|) = \Phi(\omega_j)|\tilde{\psi}(l)\rangle\langle\tilde{\psi}_j(l)| \ . \tag{9}$$

When these coherences are small (or vanish), the randomized evolution approximately (or perfectly) simulates a projective measurement onto the desired (ground) eigenstate. The error in the simulation of the measurement can then be bounded from above by the largest value of $|\Phi(\omega)|$ at frequencies ω determined by the corresponding energy differences.

The average cost of randomization (per step) is given in terms of the random variable T as $\langle|T|\rangle$, the expected value

of its absolute value. We can bound the required average cost per step from below by $\Omega(1/\Delta)$, with $0 < \Delta \leq \inf_s \Delta(s)$, by means of the following theorem:

Theorem 2: Let T be a random variable with characteristic function Φ. Then, for all ω,

$$\mathrm{cost}(T) = \langle |T| \rangle \geq \frac{1 - |\Phi(\omega)|}{|\omega|} \ .$$

Proof: From the definition of Φ we obtain

$$\begin{aligned} 1 - |\Phi(\omega)| \leq |1 - \Phi(\omega)| &\leq \int |1 - e^{i\omega t}| d\mu(t) \\ &\leq \int |\omega t| d\mu(t) = \langle |T| \rangle |\omega| \ , \end{aligned} \tag{10}$$

with μ the probability distribution of T. The Zeno effect can be achieved by simulating measurements that ensure $|\Phi(\omega)| = 0$ for $|\omega| \geq \Delta$. This gives the scaling $1/\Delta$. A possible evolution-time probability distribution that satisfies this condition can be obtained if T's probability density is proportional to $\mathrm{sinc}(\Delta t/4)^4$. The function sinc is defined as $\mathrm{sinc}(t) = \sin(t)/t$. The Fourier transform of $\Delta/(4\pi)\mathrm{sinc}(\Delta t/4)$ is the indicator function of the interval $[-\Delta/4, \Delta/4]$. Thus, by implementing the four-fold convolution of the indicator function with itself yields $\Phi(\omega) = 0$ for $|\omega| > \Delta$. The average cost $\langle |T| \rangle$ is $\mathcal{O}(1/\Delta)$. According to Thm. 2 this is optimal. Nevertheless, the probability density determined by $\mathrm{sinc}(x)^4$ has long tails and unbounded moments. Better distributions that do not suffer of this problem can be constructed as shown in Ref. [13].

III. The randomization method

The goal of the randomization method is to prepare the nondegenerate eigenstate $|\psi(1)\rangle$ of $H(1)$ by traversing a discretization of the path $|\psi(s)\rangle$. This path is determined by the family $\mathcal{H} = \{H(s)\}$. Ideally, we choose the uniform parametrization $s(l)$ discussed in Sec. II-A. Under such a parametrization, the eigenstates $|\psi(s(l))\rangle$ move at a constant unit rate along the path. We can also consider the *subuniform* parametrization $l \in [0, L'] \mapsto s'(l)$ so that the rate at which the states move is bounded by 1. The number of points in the discretization is $q \in \mathcal{O}((L')^2)$ for bounded error probability.

The randomization method uses randomized evolutions $\mathcal{R}_l^T$ to approximate the projective-measurement operations M_l at values $s(l)$. Here, $l = k\delta$ for $k = 1, \ldots, q = L'/\delta$, and δ sufficiently small. For good asymptotic behavior, we choose T's probability density to be $\propto \mathrm{sinc}(\Delta t/4)^4$ but other choices may work as well. We obtain:

Theorem 3 (Randomization method): There are choices of q and T in the randomization method such that the method outputs $|\psi(1)\rangle$ starting from $|\psi(0)\rangle$ with fidelity at least p and average cost

$$\mathcal{O}\left(\frac{(L')^2}{(1-p)\Delta}\right) \ .$$

Proof: We choose a step increment $\delta = L'/q$, with $q = \lceil 2(L')^2/(1-p) \rceil$. For this choice, Lemma 1 guarantees that, if we were to implement the projective-measurement operations exactly, as is the case for this choice of T, the error in the preparation of $|\psi(1)\rangle$ would be bounded by $(1-p)/2$. ($d \leq 1$ for the subuniform parametrization.) Suppose that the error according to Thm. 1 is bounded by ϵ. After r steps the total error is bounded by $r\epsilon$. The desired bound on the error requires $\epsilon \leq (1-p)/(2q) = (1-p)^2/(4(L')^2)$. This can be achieved at an average cost $\langle |T| \rangle$ of $\mathcal{O}(1/\Delta)$. The total cost for the procedure is $\mathcal{O}(q/\Delta)$, and substitution of the value for q yields the claimed bound.

In the most general case, we need to use the definition in Eq. (7) for L'. The cost of the randomization method in that case is $\mathcal{O}(1/\Delta^3)$ which is consistent with the one in AQC. But in many cases of interest, one can choose a parametrization $s'(l)$ that is close to uniform (up to some constant), and where $L' \in \mathcal{O}(1)$. We give an example in the following section.

IV. Examples

We give an example where the path length can be bounded independently of the bound on the gap Δ. The complexity of the randomization method in this case is $\Theta(1/\Delta)$, and is much better than $\mathcal{O}(1/\Delta^3)$ when $\Delta \ll 1$.

A. Quantum Gibbs' States

The quantum simulated annealing (QSA) algorithm discussed in Ref. [7] is designed to traverse a coherent version of the classical-state path traversed by classical simulated annealing. The quantum state path is in a Hilbert space of dimension corresponding to the size of the classical state space. The classical annealing path we consider is determined by $\pi_x(\beta) = e^{-\beta E[x]}/\mathcal{Z}$, where π_x is the probability of configuration x in the stationary (Gibbs) distribution. E is the associated energy or cost function, β is the inverse temperature, and $\mathcal{Z}$ the partition function. The corresponding path in Hilbert space is given by the *quantum Gibbs' states* $|\psi(\beta)\rangle = \sum_x \sqrt{\pi_x(\beta)}\,|x\rangle$. Note that a measurement in the computational basis samples x with probability $\pi_x(\beta)$. Since

$$|\partial_\beta \psi(\beta)\rangle = \sum_x (\langle E \rangle - E[x]) \sqrt{\pi_x}/2\,|x\rangle \ , \tag{11}$$

we obtain the following lemma.

Lemma 3: For $\beta \in (0, \beta_f)$,

$$\| \, |\partial_\beta \psi(\beta)\rangle \, \| = \sigma(\beta)/2 \ ,$$

where $\sigma(\beta)$ is the standard deviation of E at inverse temperature β. The path length satisfies $L \leq \beta_f \sigma/2$, with $\sigma = \sup_\beta \sigma(\beta)$.

That L does not depend on the spectral gap is the main reason for the success of QSA. See Ref. [7] for details.

V. Conclusion

We have described a method for state preparation in the spirit of AQC, but based exclusively on randomized evolutions. The idea is to perform a discrete sequence of projective measurements onto the desired (instantaneous) eigenstate of a given Hamiltonian or unitary path. These measurements

are induced via evolution randomization, which realizes the necessary decoherence in the eigenbasis. The complexity of the method is $\mathcal{O}(L^2/\Delta)$, with L the path length and Δ a lower bound on the gap of the Hamiltonians. Making explicit the cost dependence on L is useful for those cases where the path length is independent of the gap, allowing perhaps to show a quantum speedup for a particular algorithm. We gave an important example where this situation occurs.

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REFERENCES

[1] E. Farhi, J. Goldstone, S. Gutman, and M. Sipser, *Quantum Computation by Adiabatic Evolution*, e-print quant-ph/0001106, 2000.
[2] A. Messiah, *Quantum Mechanics*, Dover Publications, 1999.
[3] D. Aharonov, W. van Dam, J. Kempe, Z. Landau, S. Lloyd, and O. Regev, *Adiabatic Quantum Computation is Equivalent to Standard Quantum Computation*, SIAM Journal on Computing **37**, 166, 2007.
[4] S. Jansen and M.B. Ruskai, *Bounds for the Adiabatic Approximation with Applications to Quantum Computation*, e-print quant-ph/0603175, 2006.
[5] D. Lidar, A. Rezakhani, and A. Hamma, e-print arxiv:0808.2697, 2008.
[6] S.P. Jordan, *Quantum Computation Beyond the Circuit Model*, PhD thesis, e-print arxiv:0809.2307, 2008.
[7] R.D. Somma, S. Boixo, H. Barnum, and E. Knill, *Quantum Simulations of Classical Annealing Processes*, Phys. Rev. Lett. **101**, 130504, 2008.
[8] B. Misra, and E.C. Sudarshan, *The Zeno's Paradox in Quantum Theory* Journal of Mathematical Physics **18**, 756, 1977.
[9] D. Home and M.A. Whitaker, *A Conceptual Analysis of Quantum Zeno Paradox, Measurement, and Experiment*, Annals of Physics **258**, 237, 1997.
[10] H. Nazakato, M. Namiki, S. Pascazio, *Temporal Behavior of Quantum Mechanical Systems* Int. J. Mod. Phys B **10**, 247, 1996.
[11] A.M. Childs, E. Deotto, E. Farhi, J. Goldstone, S. Gutman, and A.J. Landahl, *Quantum Search by Measurement* Phys. Rev. A **66**, 032314, 2002.
[12] D. Aharonov and A. Ta-Shma, *Adiabatic Quantum State Generation and Statistical Zero Knowledge*, Proceedings of the thirty-fifth ACM symposium on Theory of computing, 20, 2003.
[13] S. Boixo, E. Knill, and R.D. Somma, *Quantum State Preparation by Phase Randomization*, e-print arxiv:0903.1652, 2009.